1.0

4.5
5.0
5.6
6.3

2.8
3.2
3.6
4.0

2.5
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2.0

1.1

1.8

1.25   1.4   1.6

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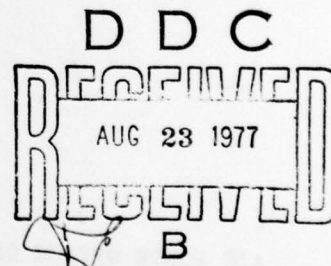DESIGN OF MINIMUM NOISE DIGITAL FILTERS USING A MIXED NORM*

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A concept of a "mixed norm" of a function is introduced and studied. It arises from the problem of design of nonrecursive digital filters.

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DESIGN OF MINIMUM NOISE DIGITAL FILTERS USING A MIXED NORM

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1. INTRODUCTION

Previous work on design of nonrecursive digital filters which are identical mathematically to uniformly-spaced linear antenna arrays [1], has minimized a quadratic error criterion, subject to inequality constraints on the maximum error in the filter approximation, at an arbitrary but finite number of points. Quadratic error and maximum error can be made to correspond, respectively, to total stopband noise power and maximum passband error at a single frequency. In the following, a weighted sum of the two error criteria will be minimized by use of quadratic programming. Furthermore, any solution to the constrained quadratic minimization problem will be shown to be a solution to a weighted sum minimization problem, and vice versa.

2. STATEMENT OF THE MATHEMATICAL PROBLEMS

The design of a low pass filter can be viewed as an approximation of the function

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq t_p, \\ 0 & \text{if } t_s \leq t \leq 0.5, \end{cases}$$

where $0 < t_p < t_s < 0.5$. The interval $[0, t_p]$ is called the passband and $[t_s, 0.5]$ is called the stopband. The approximation is done by means of

linear combinations

$$h(t) = \sum_{k=0}^{N} a_k \cos 2\pi c_k t \qquad 0.0 \le t \le 0.5$$

where the c_k's are non-negative reals, strictly increasing with k and $c_o = 0$; also the a_k's are real.

The norm which will be used is the "mixed norm" defined for every real function g, continuous on both $[0, t_p]$ and $[t_s, 0.5]$ by:

$$\| g \|^2 = \lambda \max_{0 \le t \le t_p} |g(t)|^2 + (1 - \lambda) \int_{t_s}^{0.5} \omega(t) |g(t)|^2 dt$$

with some fixed λ, $0 < \lambda < 1$, and some fixed positive continuous real-valued function ω defined on $[0, t_p] \cup [t_s, 0.5]$. The fact that this definition determines a norm is obvious except for the triangle inequality; its proof appears in the Appendix.

Our problem, $P(\lambda)$, is to find a best approximation to f, namely,

$$\text{minimize}_{a_k} \| f(\cdot) - \sum_{k=0}^{N} a_k \cos 2\pi c_k(\cdot) \| .$$

However, for computational purposes, we substitute for it another problem, $P_D(\lambda)$, obtained by discretizing the passband. $P_D(\lambda)$ is the problem

$$\text{minimize}_{a_k} \| f(\cdot) - \sum_{k=0}^{N} a_k \cos 2\pi c_k(\cdot) \|_D$$

where, for every g as above,

$$\| g \|_D^2 = \lambda \max_{i=1,\ldots,m} |g(t_i)|^2 + (1-\lambda) \int_{t_s}^{0.5} \omega(t) |g(t)|^2 dt$$

and $0 \leq t_1 < t_2 < \ldots < t_m \leq t_p$ are given numbers. $P_D(\lambda)$ reduces to a quadratic programming problem.

Each of $P(\lambda)$ and $P_D(\lambda)$ has a unique solution. Neither $\| \cdot \|$ nor $\| \cdot \|_D$ is strictly convex; however, and we omit the proof, the strict convexity of the L_2 norm can be used to show unicity of solution of each of the problems $P(\lambda)$, $P_D(\lambda)$.

In [1], a problem similar to $P(\lambda)$ was solved which also reduces to a quadratic programming problem after discretization of the passband. The problem, $P'(\epsilon)$, was

$$\underset{a_k}{\text{minimize}} \int_{t_s}^{0.5} \omega(t) \left| \sum_{k=0}^{N} a_k \cos 2\pi c_k t \right|^2 dt$$

subject to

$$1 - \epsilon \leq \sum_{k=0}^{N} a_k \cos 2\pi c_k t \leq 1 + \epsilon \qquad \text{for } 0 \leq t \leq t_p,$$

where $0 < \epsilon < 1$ and ω is as above. (Actually, in [1], only the case $c_k \equiv k$ was considered.) Let $P_D'(\epsilon)$ be the discretized version of $P'(\epsilon)$. Again, by the strict convexity of the L_2 norm, each of $P'(\epsilon)$ and $P_D'(\epsilon)$ has a unique solution.

3. EQUIVALENCE THEOREM AND PROOFS

Let $h_\lambda(t) \equiv \sum_{k=0}^{N} a_k^* \cos 2\pi c_k t$ be the unique solution to $P(\lambda)$.

Denote $\epsilon(\lambda) = \max_{0 \le t \le t_p} |1 - h_\lambda(t)|$. Let $\epsilon_D(\lambda)$ be the corresponding number for $P_D(\lambda)$. The following theorem is stated for $P(\lambda)$ and $P'(\epsilon)$, but a similar theorem is true for the discretized versions.

Remark. It is convenient to extend the problems $P(\lambda)$, $P'(\epsilon)$, $P_D(\lambda)$ and $P_D'(\epsilon)$ to the cases $\lambda = 0,1$; $\epsilon = 0,1$. Existence of solution for these problems is still true.

Theorem 1. Let f, ω, t_p, t_s and c_k, $k = 0,\ldots,N$, be fixed as above. Each $P(\lambda)$, $0 \le \lambda \le 1$, is equivalent to some $P'(\epsilon)$, $0 \le \epsilon \le 1$. That is:

(1) For each λ, $0 \le \lambda \le 1$, let h_λ be the solution of $P(\lambda)$. Then there is an ϵ, $0 \le \epsilon \le 1$, such that h_λ is the solution to $P'(\epsilon)$; and

(2) For a given ϵ, $0 \le \epsilon \le 1$, let g_ϵ be the unique solution of $P'(\epsilon)$. Then there is a λ, $0 \le \lambda \le 1$, such that g_ϵ is the solution to $P(\lambda)$.

Our first proof also applies to the corresponding discretized version of the theorem. A second proof (for that version only) follows which gives a valuable insight.

First Proof. (1) Let λ be given, $0 \le \lambda \le 1$. If h_λ, a solution to $P(\lambda)$, is not a solution to $P'(\epsilon(\lambda))$, then a solution to $P'(\epsilon(\lambda))$ would be better than h_λ in $P(\lambda)$ which, of course, is a contradiction.

(2) We will prove the second part of the theorem by showing that $\epsilon(\lambda)$ is a decreasing continuous function of λ and, in fact, maps $[0,1]$ onto itself. It is clear that $\epsilon(0) = 1$, since the unique solution of $P(0)$ is

$h_0(t) \equiv 0$. Similarly, $\epsilon(1) = 0$. Lemma 1 below is used to prove that $\epsilon(\lambda)$ is decreasing.

Lemma 1. If $0 < \lambda_1 < \lambda_2 < 1$, $a_1 < a_2$ and $b_2 < b_1$, then either

$$\text{(A)} \quad \lambda_2 a_1 + (1 - \lambda_2)b_1 < \lambda_2 a_2 + (1 - \lambda_2)b_2$$

or $\quad$ (B) $\quad \lambda_1 a_2 + (1 - \lambda_1)b_2 < \lambda_1 a_1 + (1 - \lambda_1)b_1$

Proof of Lemma 1. Either (i) $\dfrac{b_1 - b_2}{a_2 - a_1} < \dfrac{\lambda_2}{1 - \lambda_2}$ or

(ii) $\dfrac{b_1 - b_2}{a_2 - a_1} > \dfrac{\lambda_1}{1 - \lambda_1}$, since $\dfrac{\lambda_1}{1 - \lambda_1} < \dfrac{\lambda_2}{1 - \lambda_2}$. Clearly, (i) implies (A) and (ii) implies (B). Lemma 1 is proven.

If we had $\lambda_1 < \lambda_2$ with $\epsilon(\lambda_1) < \epsilon(\lambda_2)$, we would arrive at a contradiction to Lemma 1; hence, $\epsilon(\lambda)$ is decreasing for $0 < \lambda < 1$. Also $\epsilon(0) = 1 \geq \epsilon(\lambda) \geq 0 = \epsilon(1)$ for $0 < \lambda < 1$.

The continuity of $\epsilon(\lambda)$ can be established by a straightforward argument.

Second Proof (for discretized version). Convexity of the positive definite quadratic part of the objective functions and convexity of the constraint functions guarantee applicability of the Kuhn-Tucker conditions to both $P_D(\lambda)$ and $P_D'(\epsilon)$, for $0 \leq \lambda < 1$ and $0 < \epsilon \leq 1$. (See [3], pp. 20, 90.) The special cases $\lambda = 1$ and $\epsilon = 0$ give the special solution $h(t) = 1$. For the other cases, define the approximating function

$$h(\underline{a}, t) = \sum_{i=1}^{N} a_i \cos 2\pi c_i t$$

and non-negative constraint functions

$$g_{k+}(\underline{a}) = (h(\underline{a}, t_k) - 1)^+$$

$$g_{k-}(\underline{a}) = (1 - h(\underline{a}, t_k))^+$$

where $b^+ = \begin{array}{l} b,\ b > 0 \\ 0,\ b \le 0. \end{array}$

Necessary and sufficient Kuhn-Tucker conditions for $(a_1, a_2, \ldots, a_N) = \underline{a} = \underline{a}_o$ to be a solution to $P_D'(\epsilon_o)$ are

$$(*) \begin{cases} \underline{0} = \underset{\underline{a}}{grad} \|h\|_2^2 + \sum_{k=1}^{m} \sigma_k^+ \underset{\underline{a}}{grad}\, g_{k+}(\underline{a}) + \sum_{k=1}^{m} \sigma_k^- \underset{\underline{a}}{grad}\, g_{k-}(\underline{a})\Bigg|_{\underline{a}_o}, \\[2mm] \sigma_k^+ \ge 0, \qquad \sigma_k^- \ge 0, \\[2mm] \sigma_k^+ (g_{k+}(\underline{a}_o) - \epsilon_o) = 0, \\[2mm] \sigma_k^- (g_{k-}(\underline{a}_o) - \epsilon_o) = 0, \qquad\qquad k = 1, 2, \ldots, m. \end{cases}$$

If we first multiply the objective function in $P_D(\lambda)$ by $(1 - \lambda)^{-1}$, it is easy to see that $P_D(\lambda)$ is equivalent to

$$\underset{\underline{a}, \rho}{minimize}\ \ \mu_\infty \rho^2 + \|h\|_2^2 \ .$$

$$subject\ to\ \ g_{k+}(\underline{a}) \le \rho ,$$

$$g_{k-}(\underline{a}) \le \rho ,$$

where $\mu_\infty = \dfrac{\lambda}{1 - \lambda}$. The Kuhn-Tucker conditions for a global minimum for this problem at $\underline{a} = \underline{a}_o$ are

$$
(**)\left\{
\begin{aligned}
& \underline{0} = \text{grad}_{\underline{a}}\, \|h\|_2^2 + \sum_{k=1}^{m} \mu_{k+}\, \text{grad}_{\underline{a}}\, g_{k+}(\underline{a}) + \sum_{k=1}^{m} \mu_{k-}\, \text{grad}_{\underline{a}}\, g_{k-}(\underline{a})\bigg|_{\underline{a}_o'}, \\[2mm]
& 0 = 2\mu_\infty \rho - \sum_{k=1}^{m} \mu_{k+} + \mu_{k-}, \\[2mm]
& \mu_{k+} \geq 0, \qquad \mu_{k-} \geq 0, \\[2mm]
& \mu_{k+}(g_{k+}(\underline{a}_o') - \rho) \geq 0, \\[2mm]
& \mu_{k-}(g_{k-}(\underline{a}_o') - \rho) \geq 0, \qquad\qquad k = 1,2,\ldots,m.
\end{aligned}
\right.
$$

Notice that if we start with the minimum point, $\underline{a}_o$, of $P_D'(\epsilon_o)$, then set $\mu_{k\pm} = \sigma_{k\pm}$, $k = 1,2,\ldots,m$, and $\mu_\infty = \frac{1}{2}\epsilon_o^{-1} \sum_{k=1}^{m} \mu_{k+} + \mu_{k-}$, the Kuhn-Tucker conditions $(**)$ will be satisfied for $P_D(\mu_\infty)$ and hence, $\underline{a}_o$ is the minimum point for $P_D(\mu_\infty)$. Note that $\epsilon\mu_\infty = \epsilon_o$ will be the solution value of ρ.

If $\underline{a}_o'$ is the minimum point for $P_D(\lambda_o)$, then again set $\sigma_{k\pm} = \mu_{k\pm}$ and $\epsilon = \rho$ and we see that at $\underline{a}_o'$, $(*)$ also is satisfied and so $\underline{a}_o'$ is the minimum point for $P_D'(\epsilon)$. (However, this half of the theorem is immediate as shown in the beginning of the first proof, above.)

NUMERICAL EXAMPLES

We will compare the solution of a problem of type $P'(\epsilon)$ given in [1] with solutions of $P_D(\lambda)$ for various values of λ. All of the computation was done with a computer program called QPS which was available at the University of Rhode Island Computer Center. The algorithm that it uses is based on numerical methods described in [2].

We will take $c_k \equiv k$ and $\omega(t) \equiv 1$, as in the examples in [1]. The other specific quantities were:

$$m = 51, \qquad t_i = 0.002\,(i - 1), \qquad i = 1,2,\ldots,51,$$
$$t_s = 0.135,$$
$$N = 14,$$
$$\epsilon = 0.035.$$

The solution in [1] yielded a value of 0.0006 for the integral of the function over the stopband.

To illustrate the discrete version of Theorem 1 we solved numerically $P_D(\lambda)$ for various values of λ. Some numerical results are given in Table I. We see that the solution of $P_D(\lambda)$ with $\lambda = 0.165$ corresponds approximately to the solution of the problem $P_D(\epsilon)$ given in [1].

TABLE I SOLUTION OF $P_D(\lambda)$ FOR SOME λ		
λ	$\epsilon_D(\lambda)$	INTEGRAL OVER STOPBAND
0.1	0.046	0.00046
0.165	0.035^+	0.00057
0.2	0.030	0.00064
0.5	0.017	0.00080
0.8	0.014	0.00104

APPENDIX

Theorem 2. Let A and B be bounded sets of real numbers and let $\lambda \in (0,1)$ be a fixed constant. Let $\|\cdot\|_A$ be a norm (semi-norm) on $C(A)$, the set of continuous, real valued functions on A, and let $\|\cdot\|_B$ be a norm (semi-norm) on $C(B)$. Then the "mixed norm" ("mixed semi-norm") defined by

$$\|g\|^2 = \lambda \|g\|_A^2 + (1 - \lambda)\|g\|_B^2$$

is a norm (semi-norm) on $C(A) \cap C(B)$.

Proof. The only condition that is not immediate is the triangle inequality. We will show that

$$\|f + g\|^2 \leq (\|f\| + \|g\|)^2 = \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|.$$

Let f and g be in $C(A) \cap C(B)$. Then

$$\|f + g\|^2 = \lambda \|f + g\|_A^2 + (1 - \lambda)\|f + g\|_B^2$$

$$\leq \lambda \|f\|_A^2 + 2\lambda \|f\|_A \|g\|_A + \lambda \|g\|_A^2 + (1-\lambda)\|f\|_B^2 + 2(1-\lambda)\|f\|_B\|g\|_B$$

$$+ (1 - \lambda)\|g\|_B^2$$

$$= \|f\|^2 + \|g\|^2 + 2\lambda \|f\|_A \|g\|_A + 2(1 - \lambda)\|f\|_B\|g\|_B$$

$$= \|f\|^2 + \|g\|^2 + 2[\lambda^{\frac{1}{2}}\|f\|_A \cdot \lambda^{\frac{1}{2}}\|g\|_A + (1 - \lambda)^{\frac{1}{2}}\|f\|_B \cdot (1 - \lambda)^{\frac{1}{2}}\|g\|_B].$$

We apply the Cauchy-Schwarz inequality in R^2 to the quantity in the square brackets and get:

$$\|f+g\|^2 \le \|f\|^2 + \|g\|^2 + 2\left[\sqrt{\lambda\|f\|_A^2 + (1-\lambda)\|f\|_B^2} \cdot \sqrt{\lambda\|g\|_A^2 + (1-\lambda)\|g\|_B^2}\right]$$

$$= \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|.$$

Q.E.D.

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